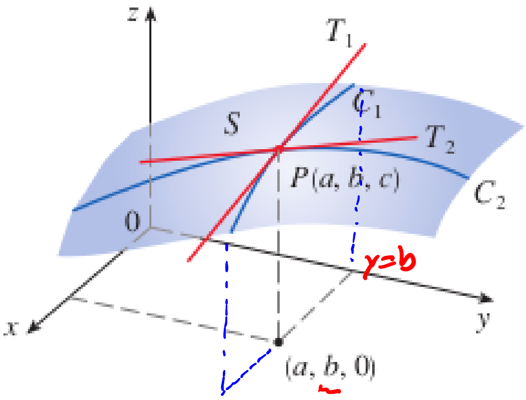
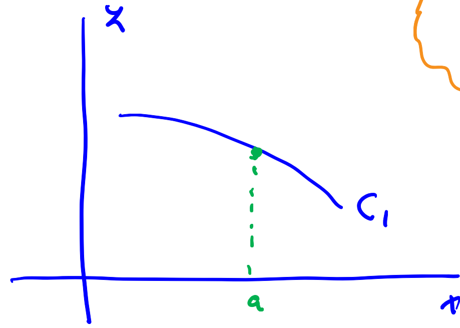
Section 14.3: PARTIAL DERIVATIVES

I. Computation of Partial Derivatives Pointwise.

In this example the slope is negative

Geometric Idea.





$$\text{slope} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$$

DEF: The partial derivative of f w.r.t. x at the point (a, b) is denoted by $f_x(a, b)$ and defined by

$$f_x(a,b) := \frac{d}{dx}\Big\{f(x,b)\Big\}\Big|_{x=a}$$

To find $f_x(a, b)$, we need to know the function $f(x, b)$ and then compute the ordinary derivative of this function at the point $x = a$.

y=2 is constant

Ex1. Let $f(x, y) = \sqrt[3]{x^3 + y^3}$. Find $f_x(1, 2)$.

x is the variable

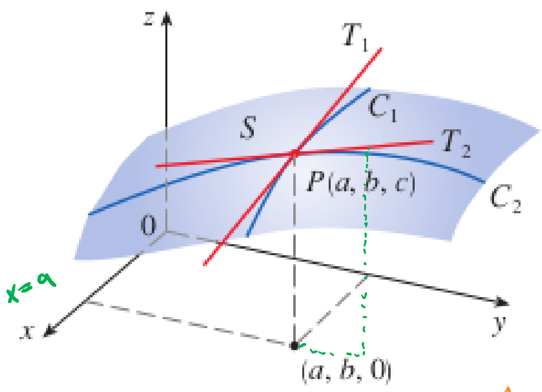
First, $f(x,2) = \sqrt[3]{x^3+8}$

then $\frac{d}{dx}\{f(x,2)\} = \frac{d}{dx}\{(x^3+8)^{1/3}\} = \frac{1}{3}(x^3+8)^{-2/3} \cdot (3x^2)$

Finally $\underbrace{\frac{d}{dx}\{f(x,2)\}\Big|_{x=1}}_{f_x(1,2)} = \underbrace{\frac{1}{3}(1+8)^{-2/3}(3(1)^2)}_{}$

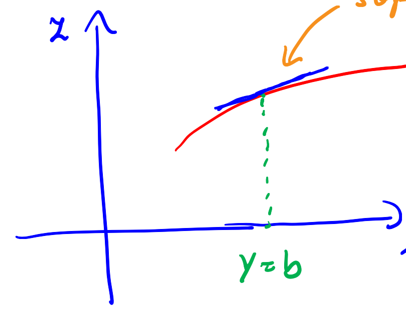
$$f_x(1,2) = \frac{1}{9^{2/3}}$$

Geometric Idea.



(x=a constant)

In the graph the slope is positive



$$\text{slope} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b)$$

with respect to

DEF: Partial derivative of f w.r.t. y at the point (a,b) is denoted by $f_y(a,b)$ and defined by

$$f_y(a,b) := \frac{d}{dy}\Big\{f(a,y)\Big\}\Big|_{y=b}$$

To find $f_y(a,b)$, we need to know the function $f(a,y)$ and then compute the ordinary derivative of this function at the point $y=b$.

1 is constant

Ex2. Let $f(x,y) = \sqrt[3]{x^3+y^3}$. Find $f_y(1,2)$.

y is the variable

First, $f(1,y) = \sqrt[3]{1+y^3}$

then $\dfrac{d}{dy}\{f(1,y)\} = \dfrac{d}{dy}\{(1+y)^{1/3}\} = \dfrac{1}{3}(1+y)^{-2/3}\cdot(3y^2)$

Finally $f_y(1,2) = \dfrac{d}{dy}\{f(1,y)\}\Big|_{y=2} = \dfrac{1}{3}(1+8)^{-2/3}(3(2)^2) = \dfrac{4}{9^{2/3}}$

II. Partial Derivatives as Functions.

Notations. If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x,y) = \frac{\partial z}{\partial x} = f_1 = D_x f = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(x,y) = \frac{\partial z}{\partial y} = f_2 = D_y f = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

How to compute f_x and f_y? Often, we can use the following procedure:

- To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

y is a constant

Ex3. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(x, y)$ and compute $f_x(1, 1)$. x is the variable

$f_x(x,y) = 3x^2 + 2xy^3 - 0$

then, $f_x(1,1) = 3(1)^2 + 2(1)(1)^3 = 5$

Ex4. Let $g(x, y) = y \sin(xy)$. Compute $\dfrac{\partial g}{\partial y}$ and $\dfrac{\partial g}{\partial y}(\pi/2, 3)$. y is the variable

$g_y(x,y) = (1)(\sin(xy)) + (y)\cos(xy)\, x = \sin(xy) + xy\cos(xy).$

$g_y(\pi/2, 3) = \sin\left(\frac{3\pi}{2}\right) + \frac{3\pi}{2}\cos\left(\frac{3\pi}{2}\right) = -1 + \frac{3\pi}{2}(0) = -1$

y=0 constant

TO-DO: Let $f(x, y) = (x^2 + y^4)e^{\sin(xy+\pi/2)}$. Find $f_x(1, 0)$. x is the variable

First $f(x,0) = x^2 e^{\sin(\pi/2)} = x^2 e$

Then $\frac{d}{dx}\{f(x,0)\} = \frac{d}{dx}\{x^2 e\} = 2xe$

Finally. $f_x(1,0) = \frac{d}{dx}\{f(x,0)\}\Big|_{x=1} = 2(1)e = 2e$

Higher-ordered Partial Derivatives: The function $f(x,y)$ has two partial derivatives, $f_x(x,y)$ and $f_y(x,y)$, one for each variable. Each of these partial derivatives has two partial derivatives, so $f(x,y)$ has four *second* partial derivatives: $f_{xx}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$, and $f_{yy}(x,y)$.

The notation $f_{xy}(x,y)$ means to find the partial derivative with respect to x *first*, and then find the partial derivative of that with respect to y.

Ex5. If $f(x,y) = x\cos y + ye^x$ find all four second partial derivatives.

$f_x(x,y) = f_x = \cos(y) + ye^x$

$f_{xx} = 0 + ye^x$; $f_{xy} = -\sin(y) + e^x$

$f_y(x,y) = f_y = -\sin(y) + e^x$

$f_{yx} = -\sin(y) + e^x$ $f_{yy} = -x\cos(y) + 0$

must be the same (on continuous functions)

Mixed Derivative Theorem: If $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are both continuous on an open disk, then they are equal at all points on that disk.

In practical terms, this means that for typical functions, the order of partial differentiation doesn't matter: $f_{xy} = f_{yx}$. This ability to proceed in different order sometimes simplifies our calculations.

Ex6. Find w_{yx} if $w = xy + \dfrac{e^y}{y^2+1}$.

Method 1

$w_y = x + \dfrac{e^y(y^2+1) - e^y(2y)}{(y^2+1)^2}$

$w_{yx} = 1 + 0 = 1$

Method 2

$w_x = y + 0$

$w_{yx} = 1 + 0 = 1$

Ex7. Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$.

Method 1

$f_y = 0 - 4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z + 0$

$f_{yxyz} = -4$

Method 2

$f_z = 0 - 2xy^2 + 0$

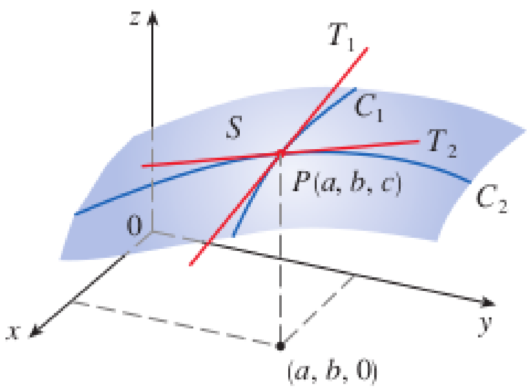
$f_{zx} = -2y^2$

$f_{zxy} = -4y$

$f_{zxyy} = -4$

Section 14.4 TANGENT PLANES AND LINEAR APPROXIMATIONS

Let S be the graph of the function $z = f(x, y)$, let (a, b) be a point in the domain of f and let $P = (a, b, f(a, b))$ be the corresponding point on the surface S. The curve C_1 is the intersection of S with the vertical plane $y = b$ while the curve C_2 is the intersection of S with the vertical plane $x = a$. T_1 is the tangent line to C_1 at the point P while T_2 is the tangent line to C_2 at the point P. Our goal is to determine the equation of the tangent plane to the surface S at the point P in terms of $f_x(a, b)$ and $f_y(a, b)$.



Definitions. Let $P_0 = (a, b)$ be a point in the domain of $z = f(x, y)$.

- The equation of the **tangent plane** to the graph of f at the point $(a, b, f(a, b))$ is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

  is called the **linearization** or **linear approximation** of f at (a, b).

Similarly, we define the linearization of $f(x, y, z)$ at a point $P_0 = (a, b, c)$ by

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - a) + f_y(P_0)(y - b) + f_z(P_0)(z - c)$$

Remarks:

- A normal vector to the tangent plane is $n = \langle f_x(a, b), f_y(a, b), -1\rangle$.
- If $(\mathbf{x}, \mathbf{y})$ is sufficiently close to (a, b) then $L(\mathbf{x}, \mathbf{y}) \approx f(\mathbf{x}, \mathbf{y})$.
- If $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is sufficiently close to (a, b, c) then $L(\mathbf{x}, \mathbf{y}, \mathbf{z}) \approx f(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Ex1. Consider $f(x,y) = x\sqrt{y}$ and the point $\underset{}{\overset{(a,b)}{(1,4)}}$. Find the equation of the tangent plane and the linearization $L(x,y)$ of the function f at the given point. Then use the linearization to estimate $f(1.2, 3.96)$.

Tools

$f_x(x,y) = \sqrt{y}$

$f_y(x,y) = \dfrac{x}{2\sqrt{y}}$

•) $f(1,4) = (1)\sqrt{4} = 2$

•) $f_x(1,4) = 2$

•) $f_y(1,4) = \dfrac{1}{2\sqrt{4}} = 1/4$

Equation of the tangent plane at $(1,4)$:

$z = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4)$

$$\boxed{z = 2 + 2(x-1) + \frac{1}{4}(y-4)}$$

Linearization of f at $(1,4)$:

$$\boxed{L(x,y) = 2 + 2(x-1) + \frac{1}{4}(y-4)}$$

Estimate $f(1.2, 3.96)$

$\approx L(1.2, 3.96)$

$f(1.2, 3.96) \approx 2 + 2(1.2-1) + \frac{1}{4}(3.96-4)$

$\approx 2 + 0.4 - 0.01$

≈ 2.39

Ex2. Approximate the value of $\sqrt{(4.01)^2 + (3.98)^2 + (2.02)^2}$ by using the linearization of a suitable function of the form $w = f(x,y,z)$.

Define $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$f(4,4,2) = \sqrt{4^2+4^2+2^2} = 6$

Tools

$f_x(x,y,z) = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$

then $f_x(4,4,2) = \dfrac{4}{6} = \dfrac{2}{3}$

$f_y(x,y,z) = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$

then $f_y(4,4,2) = \dfrac{4}{6} = \dfrac{2}{3}$

$f_z(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

then $f_z(4,4,2) = \dfrac{2}{6} = \dfrac{1}{3}$

Linearization of f at $(4,4,2)$:

$L(x,y,z) = f(4,4,2) + f_x(4,4,2)(x-4) + f_y(4,4,2)(y-4) + f_z(4,4,2)(z-2)$

$$\boxed{L(x,y,z) = 6 + \frac{2}{3}(x-4) + \frac{2}{3}(y-4) + \frac{1}{3}(z-2)}$$

Estimate $f(4.01, 3.98, 2.02) \approx L(4.01, 3.98, 2.02)$

$\approx 6 + 2/3(0.01 + 2/3(-0.02) + 1/3(0.02)$

≈ 6 ✓

Practice: what about $f(4.01, 3.98, 2.03)$?

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$dy = L(a+h) - L(a)$

Differentials

Let $f(x,y)$ be a function defined nearby the point (a,b). Suppose $f_x(a,b)$ and $f_y(a,b)$ both exist. If we move from (a,b) to a point $(a+dx, b+dy)$ nearby, show that the resulting change in the linearization of f is given by

$$df = L(a+dx, b+dy) - L(a,b) = f_x(a,b)dx + f_y(a,b)dy$$

$$L(a+dx, b+dy) = f(a,b) + f_x(a,b)(\cancel{a}+dx-\cancel{a}) + f_y(a,b)(\cancel{b}+dy-\cancel{b})$$

$$= f(a,b) + f_x(a,b)\,dx + f_y(a,b)\,dy$$

so $df = L(a+dx, b+dy) - L(a,b) = \cancel{f(a,b)} + \underline{f_x(a,b)dx + f_y(a,b)dy} - \cancel{f(a,b)}$

Let $\Delta f = f(a+dx, b+dy) - f(a,b)$. Since $f(a,b) = L(a,b)$ and $L(a+dx, b+dy) \approx f(a+dx, b+dy)$ we have that

$$\Delta f \approx df = f_x(a,b)dx + f_y(a,b)dy$$

Thus, the change in the linearization df can be used to approximate the actual change in the function Δf.

Similarly, if we move from $P_0 = (x_0, y_0, z_0)$ to a point (x_0+dx, y_0+dy, z_0+dz) nearby, the resulting change in the linearization of f

$$df = f_x(x_0,y_0,z_0)dx + f_y(x_0,y_0,z_0)dy + f_z(x_0,y_0,z_0)dz$$

is a good approximation to $\Delta f = f(x_0+dx, y_0+dy, z_0+dz) - f(x_0,y_0,z_0)$.

DEF: The differential (or total differential) of $f(x,y)$ is defined to be

$$df = f_x(x,y)dx + f_y(x,y)dy.$$

DEF: The differential (or total differential) of $f(x,y,z)$ is defined to be

$$df = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz$$

Note: For functions in two variables, sometimes the notation dz is used in place of df.

Ex3. Let $z = f(x, y) = x^2 + 3xy - y^2$. Find the differential dz in general. If x changes from 2 to 2.05 and y changes from 3 to 2.96 compare the values of Δz and dz.

In general, $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$$\boxed{dz = (2x+3y)\,dx + (3x-2y)\,dy}$$

when $x=2$, $dx = 0.05$ $\quad y=3$, $dy = -0.04$

then $dz = (2(2)+3(3))(0.05) + \cancel{(3(2)-2(3))(-0.04)}$

$6 - 6$

0

$$\boxed{dz = 13\left(\frac{5}{100}\right) = \frac{65}{100} = 0.65}$$

$\Delta z = f(2.05, 2.96) - f(2,3)$

$= ((2.05)^2 + 3(2.05)(2.96) - (2.96)^2) - (2^2 + 3(2)(3) - (3)^2)$

$= 0.6449$

Ex4. Suppose that T is to be calculated from the formula $T = x(e^y + e^{-y})$, where x and y are found to be 2 and $\ln 2$ with maximum possible errors of $|dx| = 0.1$ and $|dy| = 0.02$. Use differentials to estimate the maximum error in the calculated value of T.

$\Delta T \approx dT$

In general: $dt = T_x(x,y)\,dx + T_y(x,y)\,dy$

$dt = (e^y + e^{-y})\,dx + (x(e^y - e^{-y}))\,dy$

$e^{-\ln(2)} = e^{\ln(2^{-1})}$

$= 2^{-1}$

$= \frac{1}{2}$

when $x=2$ and $\ln(2)$:

$dt = (2 + 1/2)\,dx + (2(2 - 1/2))\,dy$

$dt = 5/2\,dx + 3dy$

then

$|dt| = \left|\frac{5}{2}dx + 3dy\right| \le \left|5/2\,dx\right| + |3dy|$

thus $|dt| \le 5/2\,|dx| + 3|dy|$

so $|dt| \le \frac{5}{2}(0.1) + 3(0.02) = 5/2(1/10) + 3\left(\frac{2}{100}\right) = \frac{31}{100}$

$|dt| \le \frac{31}{100} = 0.31$